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HYDROMAGNETIC STABILITY OF STELLAR ATMOSPHERES

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SUMMARY

The effect of a horizontal magnetic field on the equilibrium of an inviscid, perfectly conducting, stratified plasma under uniform rotation is investigated. The equations of the problem are established when both the density and magnetic field vary with the vertical distance. Then special density distributions are studied, first a layer of stratified plasma and secondly, a configuration of incompressible fluid topped by a stratified compressible fluid. Criteria of instability are derived.

CONTENTS

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Satya P. Talwar†
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INTRODUCTION

The problem of the hydromagnetic stability of a conducting fluid of variable density, subject to a gravitational field, is of considerable interest in plasma confinement and astrophysical applications. It has been investigated in recent years by several workers (References 1-6). In particular, the problem is relevant to the stability of stellar atmospheres in a magnetic field. The present paper investigates the effect of a horizontal magnetic field on the equilibrium of an inviscid, perfectly conducting, *compressible* fluid of variable density. The entire configuration is assumed to partake in a uniform rotation, in view of the important role which Coriolis forces play in various astrophysical situations. We shall establish the general equations of the problem, assuming that both the density and the horizontal magnetic field vary in the upward direction. Two special cases of density distribution will be considered, a layer of stratified plasma and a configuration of incompressible fluid which is topped by a stratified compressible fluid.

EQUATIONS OF THE PROBLEM

Consider a system of cartesian axes with the z-axis in the vertical direction. Suppose that a horizontal magnetic field H_0 (stratified upwards) exists along the x direction in a compressible fluid with a variable density in the upward direction. The configuration is assumed to rotate uniformly with an angular velocity Ω about the z-axis.

The equation of motion with respect to a rotating frame of reference is written‡

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\, \mathbf{u} = -\nabla p + \mathbf{g}\rho + \frac{\mu}{4\pi}\left[(\nabla \times \mathbf{H}) \times \mathbf{H}\right] + 2\rho(\mathbf{u} \times \mathbf{\Omega}) \quad , \tag{1}$$

where $\mathbf{u}$ and $\mathbf{H}$ denote the velocity and the magnetic field vectors and p , ρ, and $\mathbf{g}$ denote the

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‡Equation 1 should, strictly speaking, include the centrifugal force term which contributes both in the equilibrium and the perturbed state. The results obtained, though exact for zero rotation, hold only for small scale lengths in the presence of small rotation.

pressure, density at a point, and acceleration due to gravity, with the component $-g$ in the z-direction (g even may denote the net acceleration downward in the case of an acceleration imposed in addition to gravity). The symbol μ represents the permeability of the medium.

The equation for the continuity of matter is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 . \tag{2}$$

The adiabatic equation (conduction of heat neglected) is

$$\frac{1}{p}\frac{dp}{dt} = \frac{\gamma}{\rho}\frac{d\rho}{dt} , \tag{3}$$

γ denoting the ratio of specific heats. Furthermore, for a perfectly conducting fluid

$$\frac{\partial \mathbf{H}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{H}) , \tag{4}$$

and finally we have the equation

$$\nabla \cdot \mathbf{H} = 0 . \tag{5}$$

The equilibrium state is characterized by $\mathbf{u} = 0$. For investigating its stability, we shall consider the effect of a small velocity field disturbance $\mathbf{u}$, with components u, v, and w in the x, y, and z directions, respectively. Then

$$\begin{aligned} \rho &= \rho_0(z) + \delta\rho\,(x, y, z, t) , \\ p &= p_0(z) + \delta p\,(x, y, z, t) , \\ \mathbf{H} &= \mathbf{H}_0(z) + \mathbf{h}\,(x, y, z, t) , \end{aligned} \tag{6}$$

where $\delta\rho$, δp, and $\mathbf{h}$ denote perturbations of the first order of smallness, so that powers higher than the first and their mutual products can be neglected. It is assumed that the components of the disturbance vary with x, y, z, and t as some function of z times $\exp\left(ik_x x + ik_y y + nt\right)$ where k_x and k_y denote the horizontal wave numbers of the harmonic disturbance and n determines the stability of the configuration. Thus the perturbation equations can be written

$$n\rho_0 u = -ik_x \delta p + \frac{\mu}{4\pi} h_z D H_0 + 2\rho_0 v \Omega , \tag{7}$$

$$n\rho_0 v = -ik_y \delta p + \frac{\mu}{4\pi} H_0 \left(ik_x h_y - ik_y h_x\right) - 2\rho_0 u\Omega , \tag{8}$$

$$n\rho_0 w = -D\delta p - g\delta\rho - \frac{\mu H_0}{4\pi}\left(Dh_x - ik_x h_z + h_x \frac{DH_0}{H_0}\right), \tag{9}$$

$$n\delta\rho = -\rho_0 \nabla\cdot\mathbf{u} - wD\rho_0 , \tag{10}$$

$$n\delta p = w\left(g\rho_0 + \frac{\mu H_0}{4\pi} DH_0\right) - c^2\rho_0\nabla\cdot\mathbf{u} , \tag{11}$$

$$nh_x = -\left(H_0 ik_y v + wDH_0 + H_0 Dw\right) , \tag{12}$$

$$nh_y = H_0 ik_x v , \tag{13}$$

$$nh_z = H_0 ik_x w , \tag{14}$$

$$ik_x h_x + ik_y h_y + Dh_z = 0 . \tag{15}$$

D stands for d/dz and c for the velocity of sound $(\gamma p_0/\rho_0)^{\frac{1}{2}}$ for the medium. Equation 11 is obtained from Equation 3 by making use of Equation 10 and of Equation 1 for the static configuration.

We now shall derive an equation for the component w of the velocity vector. Equations 7 and 8 are multiplied by ik_x and ik_y, respectively, and then added to give

$$n\rho_0(\nabla\cdot\mathbf{u} - Dw) - 2\rho_0\Omega\xi - \frac{\mu}{4\pi}\left[ik_x h_z DH_0 + ik_y H_0\left(ik_x h_y - ik_y h_x\right)\right] = k^2\,\delta p , \tag{16}$$

where

$$k^2 = k_x^{\,2} + k_y^{\,2} ,$$

$$\xi = ik_x v - ik_y u .$$

The perturbation δp is eliminated from Equations 9 and 16 by multiplying the former by k^2, operating on the latter by D, and adding. By using Equation 10

$$\begin{aligned}
&-n\rho_0\left(D^2 - k^2\right)w - D\rho_0\left(nDw + \frac{gk^2}{n} w\right) - \nabla\cdot\mathbf{u}\left(\frac{gk^2}{n}\rho_0 - nD\rho_0\right) - 2\Omega D\left(\rho_0\xi\right) \\
&\quad + n\rho_0 D\nabla\cdot\mathbf{u} - \frac{\mu}{4\pi} D\left[ik_x h_z DH_0 + ik_y H_0\left(ik_x h_y - ik_y h_x\right)\right] \\
&\quad + \frac{\mu}{4\pi} H_0 k^2\left(Dh_x - ik_x h_z\right) + \frac{\mu}{4\pi} k^2 h_x DH_0 = 0 .
\end{aligned} \tag{17}$$

The magnetic terms in Equation 17, on using Equations 13-15, reduce to

$$- \frac{\mu}{4\pi} \frac{k_x{}^2 H_0{}^2}{n} \left[\left(D^2 - k^2\right) w + \frac{2DH_0}{H_0} Dw \right] . \tag{18}$$

Thus Equation 17 becomes

$$\left(1 + \frac{k_x{}^2 V^2}{n^2}\right) \left(D^2 - k^2\right) w + \frac{D\rho_0}{\rho_0} \left(\frac{gk^2}{n^2} w + Dw\right) + \nabla \cdot \mathbf{u} \left(\frac{gk^2}{n^2} - \frac{D\rho_0}{\rho_0}\right)$$

$$+ \frac{2\Omega}{n\rho_0} D\left(\rho_0 \xi\right) - D\nabla \cdot \mathbf{u} + 2 \frac{k_x{}^2 V^2}{n^2} \frac{DH_0}{H_0} Dw = 0 , \tag{19}$$

where V denotes the Alfvén velocity $\left(\mu H_0{}^2/4\pi\rho_0\right)^{\frac{1}{2}}$. The expression for $\nabla \cdot \mathbf{u}$ now can be obtained from Equation 11. After some algebraic calculations including the use of Equations 12-16, it is found that

$$\nabla \cdot \mathbf{u} \left(1 + \frac{n^2}{c^2 k^2} + \frac{k_y{}^2 V^2}{c^2 k^2}\right) = \frac{n^2}{c^2 k^2} Dw + \frac{gw}{c^2} + \frac{\xi}{c^2 k^2} \left(ik_x ik_y V^2 + 2\Omega n\right) . \tag{20}$$

The expression for ξ can be obtained by multiplying Equations 7 and 8 by $-ik_y$ and ik_x, respectively, and adding. With the use of Equations 12-15,

$$\left(n^2 + k_x{}^2 V^2\right) \xi = 2\Omega n Dw + \nabla \cdot \mathbf{u} \left[ik_x ik_y V^2 - 2\Omega n\right] . \tag{21}$$

Combining Equations 20 and 21 finally gives

$$\nabla \cdot \mathbf{u} \left[\left(n^2 + k_x{}^2 V^2\right)\left(n^2 + c^2 k^2\right) + n^2 \left(4\Omega^2 + k_y{}^2 V^2\right)\right] = \left(n^2 + k_x{}^2 V^2\right)\left(n^2 Dw + gk^2 w\right) + 2\Omega n Dw \left(ik_x ik_y V^2 + 2\Omega n\right) \tag{22}$$

By using Equations 21 and 22 ξ and $\nabla \cdot \mathbf{u}$ may be eliminated from Equation 19. Thus the equation for component w of the velocity perturbation vector may be determined. However, for simplification it will be assumed that the velocity of sound c and the local Alfvén velocity V are independent of the vertical coordinate z (i.e., the horizontal field is so stratified in the z direction that the ratio H_0^2/ρ_0 remains constant throughout the stratified plasma). It should be mentioned that the equation determining the initial state of the configuration of an isothermal ideal gas can be satisfied with the stratification formulas

$$\left.\begin{aligned} \rho_0(z) &= \rho_1 e^{\beta z} , \\ H_0(z) &= H_1 e^{\beta z/2} , \end{aligned}\right\} \tag{23}$$

where the constant β is given by

$$\beta = \frac{-g}{\frac{V^2}{2} + \frac{c^2}{\gamma}} \; . \tag{24}$$

With the mentioned simplification, Equation 19 determining w is (after some added simplification)

$$\begin{aligned} & D^2 w + \beta D w - \left(\left(n^2 + k_x^2 V^2\right) - g\beta + \frac{g\left(n^2 + k_x^2 V^2\right)\left\{-gk^2 + n^2\beta\left[1 - \frac{2\Omega}{n\left(n^2 + k_x^2 V^2\right)}\left(ik_x ik_y V^2 - 2\Omega n\right)\right]\right\}}{\left\{\left(n^2 + k_x^2 V^2\right)\left(n^2 + k^2 c^2\right) + n^2\left(4\Omega^2 + k_y^2 V^2\right)\right\}} \right) \\ & \cdot \left\{ \left(n^2 + k_x^2 V^2\right) + \frac{4\Omega^2 n^2}{\left(n^2 + k_x^2 V^2\right)} - \left[n^2 - \frac{2\Omega n}{\left(n^2 + k_x^2 V^2\right)}\left(ik_x ik_y V^2 - 2\Omega n\right)\right] \right. \\ & \left. \cdot \frac{n^2\left(n^2 + k_x^2 V^2\right) + 2\Omega n \left(2\Omega n + ik_x ik_y V^2\right)}{\left(n^2 + k_x^2 V^2\right)\left(n^2 + k^2 c^2\right) + n^2\left(4\Omega^2 + k_y^2 V^2\right)} \right\}^{-1} wk^2 = 0 \; . \end{aligned} \tag{25}$$

Now two special cases will be considered–the hydromagnetic stability of (1) a continuously stratified layer of plasma, and (2) a configuration of two superposed fluids, the lower a homogeneous liquid and the upper a stratified gas.

HYDROMAGNETIC STABILITY OF A ROTATING STRATIFIED LAYER OF PLASMA

Here the special case will be considered in which an inviscid, infinitely conducting layer of plasma is confined between two perfectly conducting, rigid horizontal walls at $z = 0$ and $z = l$, the fluid having the density and field variations given by Equation 23 so that the sound speed c and the Aflvén velocity V are independent of height.

The solution of Equation 25 is of the form,

$$w(z) = Ae^{m_1 z} + Be^{m_2 z} \; , \tag{26}$$

where A and B are constants to be determined from the boundary conditions and m_1 and m_2 are the roots of Equation 25. The boundary conditions are that the normal components of velocity and of the magnetic field should vanish at both $z = 0$ and $z = l$. Thus,

$$\left.\begin{aligned} B &= -A\,, \\ e^{(m_1 - m_2)l} &= 1\,, \end{aligned}\right\} \tag{27}$$

and so

$$(m_1 - m_2)l = 2\alpha i\pi\,, \tag{28}$$

where α is an integer. Equation 28, with Equation 25, enables us to write the dispersion formula in the general case, which agrees when $c \to \infty$ with the one obtained in an earlier paper (Reference 6). However, the dispersion relations for three special cases will be considered, namely (1) no magnetic field, both k_x and k_y nonzero, (2) $k_x = 0$, $k_y \neq 0$ (interchange perturbations) in the presence of a magnetic field, and (3) $k_y = 0$, $k_x \neq 0$, again in the presence of a magnetic field.

Compressible Rotating Stratified Configuration with No Magnetic Field

This is purely a gas-dynamical case and the dispersion relation is

$$n^4 + n^2\left[k^2c^2(1 + E) + 4\Omega^2\right] + k^2\left(4\Omega^2Ec^2 - g^2 - g\beta c^2\right) = 0 \tag{29}$$

where $E = \left(4\alpha^2\pi^2 + \beta^2l^2\right)/4k^2l^2$. The parameter β in the above equation is given by (cf. Equation 24)

$$\beta = -g\gamma/c^2\,, \tag{30}$$

and it follows that the last term in Equation 29 is positive. Thus, the configuration is thoroughly (for all k) stable as would also be the case if the rotation were absent. Therefore Equation 29 describes the propagating waves in a uniformly rotating, stratified, compressible fluid in a gravity field, with constant sound speed c. It can be easily verified from Equation 29 that rotation adds to the stability of the configuration, the effect being more pronounced on long wave perturbations. It follows that

$$\left.\begin{aligned} n_1^2 &= -4\Omega^2\,, \\ n_2^2 &= -\left(\frac{g^2\gamma^2}{4c^2} + \frac{\alpha^2\pi^2}{l^2}c^2\right) \end{aligned}\right. \tag{31}$$

for $k \to 0$, and

$$n^2 = -\frac{g^2(\gamma - 1)}{c^2} \tag{32}$$

for $k \to \infty$.

A Stratified Plasma with a Horizontal Field for Interchange Perbations Perturbations $k_x = 0, k_y = k$

For this case the dispersion relation can be written

$$n^4 + n^2\left[k^2(c^2 + V^2)(1 + E) + 4\Omega^2\right] + k^2\left[4\Omega^2 E(c^2 + V^2) - g\beta(c^2 + V^2) - g^2\right] = 0 . \tag{33}$$

Again, since $\beta = -g/\left[(V^2/2) + (c^2/\gamma)\right]$, it readily follows that the roots (n^2) of the above equation are both negative real, implying stability.

It is seen from the above equation that

$$\begin{aligned} n_1^2 &= -4\Omega^2 , \\ n_2^2 &= -(c^2 + V^2)\left[\frac{\alpha^2\pi^2}{l^2} + \frac{g^2}{4\left(\frac{c^2}{\gamma} + \frac{V^2}{2}\right)^2}\right] \end{aligned} \tag{34}$$

for $k \to 0$, and

$$n^2 = -g^2\left[\left(\frac{c^2}{\gamma} + \frac{V^2}{2}\right)^{-1} - (c^2 + V^2)^{-1}\right] \tag{35}$$

for $k \to \infty$.

Thus, a rotating layer of plasma with constant Alfvén and sound speeds is stable for interchange perturbations. However, the perturbations characterized by a nonvanishing k_x may bring about instability. Therefore, the stability of this configuration now will be investigated for perturbations with $k_y = 0$, $k_x = k$.

A Stratified Plasma with a Horizontal Field for $k_y = 0, k_x = k$

For a uniformly rotating plasma layer confined between two rigid walls and subject to a horizontal magnetic field and vertical gravity field, so that the Alfvén and sound speeds are constant, the dispersion relation is

$$\begin{aligned} n^6 &+ n^4k^2\left[(c^2 + V^2)(1 + E) + V^2 + 4\Omega^2/k^2\right] \\ &+ n^2k^2\left\{k^2V^2(1 + E)(2c^2 + V^2) - (g\beta c^2 + g^2) + 4\Omega^2\left[V^2 + (c^2 + V^2)E\right]\right\} \\ &+ k^4V^2\left[k^2V^2c^2(1 + E) - (g\beta c^2 + g^2)\right] = 0 , \end{aligned} \tag{36}$$

where β is given by Equation 24.

If the rotation is zero, Equation 36 reduces to

$$n^4 + n^2k^2(1 + E)\left(c^2 + V^2\right) + k^2\left[k^2(1 + E)\, c^2V^2 - \left(g^2 + g\beta c^2\right)\right] = 0\,. \tag{37}$$

Equation 37 admits real roots for n^2 which are both negative, implying stability, if the coefficient of the last term is positive. For this to occur we should have either (1) $\left(g^2 + g\beta c^2\right) < 0$ characterizing the medium, or (2) the wave number of perturbation greater than a critical value $k_* \,(=2\pi/\lambda_*)$ given by

$$k_* = \left[\frac{g\beta c^2 + g^2}{V^2c^2} - \left(\frac{\alpha^2\pi^2}{l^2} + \frac{\beta^2}{4}\right)\right]^{\frac{1}{2}} \tag{38}$$

Therefore, the instability in the configuration arises only for $k < k_*$.

It follows that a static unbounded, semi-infinite plasma is stable for all modes of perturbation if the prevailing horizontal field (at $z = 0$) is less than a certain critical value H_*, obtained by setting $k_* = 0$ in Equation 38. We obtain

$$\frac{H_*^2}{2\pi NKT} = (3\gamma - 4)\left\{1 \pm \left[1 + \frac{16(\gamma - 1)}{(3\gamma - 4)^2}\right]^{\frac{1}{2}}\right\}. \tag{39}$$

N and T denote the particle density and the temperature of the medium, and K is Boltzmann's constant. Table 1 gives the values of the critical field above which instability exists for $k < k_*$ for a few physical situations, with $\gamma = 5/3$. Figure 1 gives a plot of the critical wavelength in units of c^2/g (above which the configuration is unstable) vs. c^2/V^2 for an indefinitely extended plasma. The effect of boundaries (finite l) is to diminish k_*, and therefore is stabilizing. It is interesting to note from Figure 1 that there exists a minimum value of critical wavelength, given by $\lambda_* \approx 18\, c^2/g$ so that wavelengths shorter than this value cannot be destabilized by a magnetic field. The results for the calculation of the growth rate of a mode of perturbation as a function of the magnetic field in the unstable situation of an unbounded plasma are shown in Figure 2 where n/kc is plotted against V^2/c^2 for $B\left(=g^2/4k^2c^4\right)$ equal to 4 and 6. The curves show that (1) an increase in gravity increases the growth rate of instability of a prescribed wavelength perturbation (in the range λ_* to ∞) for any magnetic field, (2) for a certain magnetic field the growth rate at the given wavelength is maximum, and (3) this magnetic field and the corresponding growth rate both increase with the gravity field.

Table 1

The Critical Fields for Several Cases.

Physical Situation	Critical field H_* (gauss)
Interplanetary matter	$\approx 2 \times 10^{-5}$
Planetary nebulae	$\approx 2 \times 10^{-3}$
Solar corona	$\approx 6 \times 10^{-2}$
Solar chromosphere	≈ 10
Laboratory plasma	$\approx 6 \times 10^{4}$

Again it follows from the dispersion relation (Equation 37) that a mode of maximum instability exists in the unstable range 0 to k_* of the wave numbers. To compute the

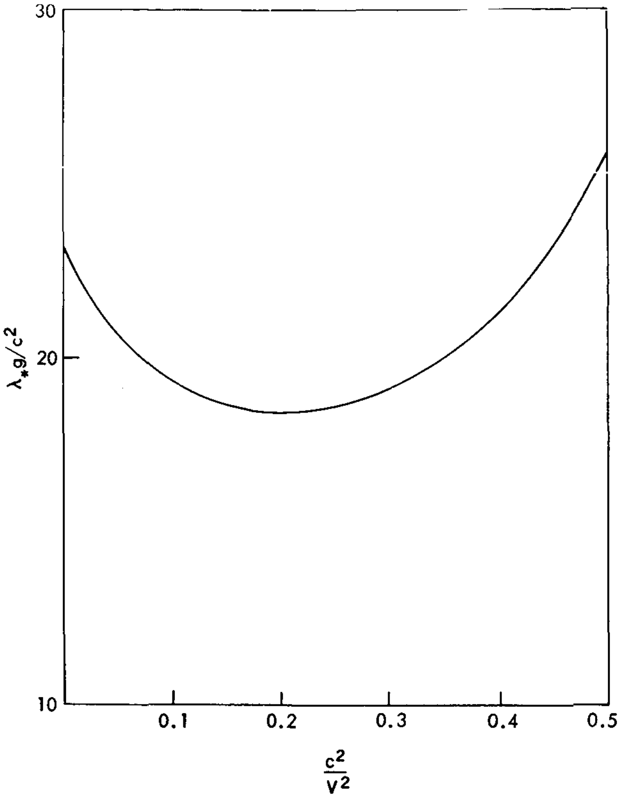

Figure 1—The critical wavelength λ_* (in units of g/c^2) as a function of c^2/V^2 for a semi-infinite stratified plasma.

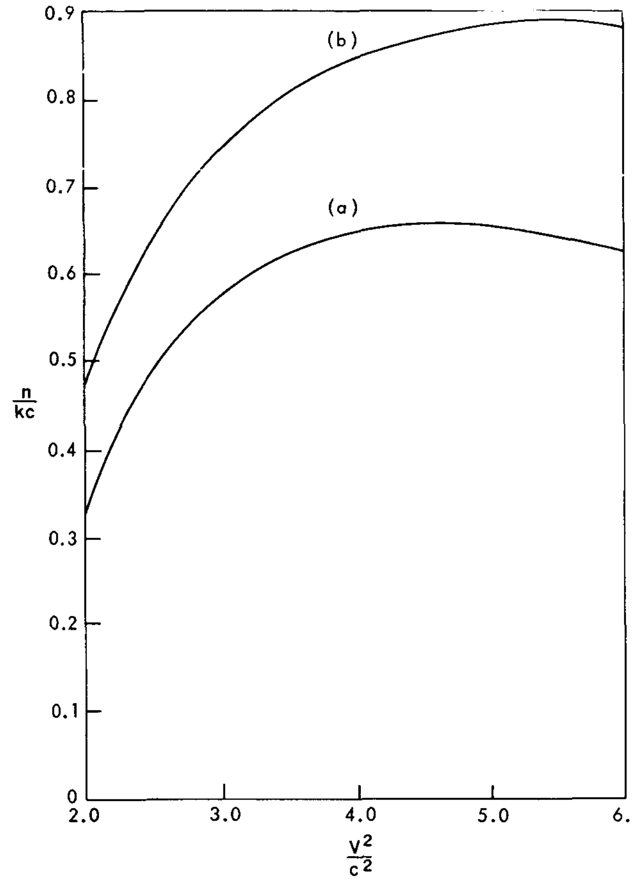

Figure 2—The plot of n/kc against V^2/c^2 for B ($=g^2/4k^2c^4$) equal to 4 and 6 (curves a and b respectively).

growth rate as a function of wave numbers let us write the dispersion relation (Equation 37) for a finite layer of plasma in dimensionless form, by measuring n and k respectively in units of $\pi V/l$ $\sec^{-1}$ and π/l cm^{-1}. We have

$$n^4 + n^2(1 + M)\left(k^2 + \alpha^2\right) k^2\left[M\left(k^2 + \alpha^2\right) - b(M + a)\right] = 0 , \tag{40}$$

where $M = c^2/V^2$, $b = g\beta l^2/\pi^2 V^2$, and $a = g/\beta V^2$ are pure numbers. It is assumed that $|\beta l| << 1$, which implies that the total change in density between $z = 0$ and $z = l$ is much less than the average density. The calculation of the positive root of Equation 40 has been carried out for $\alpha = 1$, $a = -10$, $b = -1$, and $M = 1.0$ and 1.5. The results are presented in Figure 3. Curves of n against k show clearly that the maximum growth rate, n_m, and the corresponding wave number, k_m, increase with a decrease in M. Since the decrease in M is associated with a smaller temperature of the medium it may be concluded that, for a given a and b, the effect of a decrease in temperature is to increase both the growth rate and the wave number of the mode of maximum instability.

HYDROMAGNETIC STABILITY OF AN INCOMPRESSIBLE FLUID TOPPED BY A STRATIFIED COMPRESSIBLE FLUID, WITH UNIFORM ROTATION

In this section we shall investigate the effect of a horizontal magnetic field on the equilibrium of a configuration of two superposed (immiscible) fluids of semi-infinite extent which are uniformly rotating, inviscid, and infinitely conducting. The fluid occupying the region $z < 0$ is incompressible and homogeneous with density ρ_2 and permeated by a uniform magnetic field H_2 along the x direction. The fluid in the region $z > 0$ is stratified vertically, with regard to the density and the magnetic field along the x-axis, in such a manner that the entire fluid is characterized by a speed of sound c_1 and Aflvén velocity V_1 independent of height. Let ρ_1 and H_1 denote the density and the field for the upper fluid at the level $z = 0$. Thus a current sheet exists at the common interface $z = 0$.

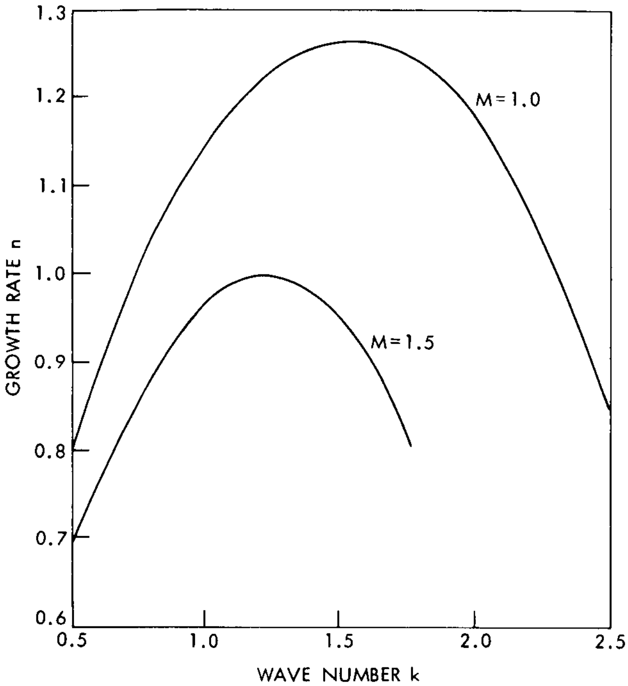

Figure 3—The growth rate n (measured with the unit $\pi V/l$ $\sec^{-1}$) is plotted as a function of wave number k (measured with the unit π/l cm^{-1}) for $\alpha = 1$, taking $b(= g\beta l^2/\pi^2 V^2) = -1$, $a\ (= g/\beta V^2) = -10$ and $M\ (= c^2/V^2) = 1.0$ and 1.5.

The perturbation equation for the incompressible fluid $(z < 0)$ is easily derived from Equation 25 by setting $\beta = 0$ and $c = \infty$,

$$D^2 w_2 - k^2 w_2 \frac{(n^2 + k_x^2 V_2^2)^2}{(n^2 + k_x^2 V_2^2)^2 + 4\Omega^2 n^2} = 0\,, \tag{41}$$

where the subscript "2" stands for the lower fluid $(z < 0)$.

The solution of Equation 41 vanishing at $z = -\infty$ is of the form

$$w_2(z) = De^{\eta z}\,, \tag{42}$$

where D is a constant and η, regarded as having a positive real part, is given by

$$\eta^2 = k^2 \left[1 + \frac{4\Omega^2 n^2}{(n^2 + k_x^2 V_2^2)^2}\right]^{-1}. \tag{43}$$

The perturbation equation for the top, compressible fluid is the same as Equation 25 whose solution under the boundary condition that $w_1 \to 0$ as $z \to \infty$ is

$$w_1(z) = Ae^{m_1 z} . \tag{44}$$

Here m_1, regarded as having a negative real part, is

$$m_1 = \frac{\beta}{2}\left[\left(1 + \frac{4k^2\epsilon}{\beta^2}\right)^{\frac{1}{2}} - 1\right] , \tag{45}$$

where β is as given by Equation 24 and ϵ is the coefficient of wk^2 Equation 25.

The boundary conditions to be satisfied at the common interface are:

1. The normal component of velocity is continuous; i.e., in the present approximation $w(z)$ is continuous at $z = 0$, leading to $A = D$.

2. The normal component of the magnetic field is continuous at the physical interface. This condition automatically follows, as can be readily verified.

3. The pressure should be continuous across the interface, which requires that

$$\delta p_1 - \delta p_2 + g(\rho_2 - \rho_1)\xi + \frac{\mu}{4\pi}\left(H_1 h_{x_1} - H_2 h_{x_2}\right) = 0 , \tag{46}$$

where ξ denotes the displacement of the interface given by $(w_1)_{z=0/m}$, and the subscripts 1 and 2 refer to the upper and the lower fluids, respectively.

On making use of Equations 12, 16, 21, and 22, we obtain for the upper, compressible fluid, at $z = 0$,

$$k^2\delta p_1 + \frac{\mu k^2}{4\pi} H_1 h_{x_1} = -\frac{m_1 A\rho_1}{n\left(n^2 + k_x^2 V_1^2\right)}\left[\left(n^2 + k_x^2 V_1^2\right)^2 + 4\Omega^2 n^2\right]$$

$$+ n\rho_1(\nabla \cdot \mathbf{u})_{z=0}\left[1 - \frac{2\Omega\left(ik_x ik_y \frac{V_1^2}{n} - 2\Omega\right)}{n^2 + k_x^2 V_1^2}\right] \tag{47}$$

where

$$(\nabla \cdot \mathbf{u})_{z=0} = \frac{A\left[\left(n^2 m_1 + gk^2\right)\left(n^2 + k_x^2 V_1^2\right) + 2\Omega n m_1\left(2\Omega n + ik_x ik_y V_1^2\right)\right]}{\left(n^2 + k_x^2 V_1^2\right)\left(n^2 + k^2 c_1^2\right) + n^2\left(4\Omega^2 + k_y^2 V_1^2\right)} . \tag{48}$$

Similarly, for the lower, incompressible fluid at $z = 0$.

$$k^2\delta p_2 + \frac{\mu}{4\pi} k^2 H_2 h_{x_2} = -\frac{A\eta\rho_2}{n\left(n^2 + k_x^2 V_2^2\right)}\left[\left(n^2 + k_x^2 V_2^2\right) + 4\Omega^2 n^2\right] . \tag{49}$$

Thus, by utilizing Equations 42, 44, 47, and 48 and the above boundary conditions, the dispersion relation for the case under investigation is found to be

$$gk^2(\rho_2 - \rho_1) + \left\{ \frac{\rho_2 \eta}{n^2 + k_x^2 V_2^2} \left[\left(n^2 + k_x^2 V_2^2\right)^2 + 4\Omega^2 n^2 \right] - \frac{m_1 \rho_1}{n^2 + k_x^2 V_1^2} \left[\left(n^2 + k_x^2 V_1^2\right)^2 + 4\Omega^2 n^2 \right] \right\}$$

$$+ n^2 \rho_1 \left[1 - \frac{2\Omega\left(ik_x ik_y \frac{V_1^2}{n} - 2\Omega\right)}{n^2 + k_x^2 V_1^2} \right] \left[\frac{\left(n^2 m_1 + gk^2\right)\left(n^2 + k_x^2 V_1^2\right) + 2\Omega n m_1 \left(2\Omega n + ik_x ik_y V_1^2\right)}{\left(n^2 + k_x^2 V_1^2\right)\left(n^2 + k^2 c_1^2\right) + n^2 \left(4\Omega^2 + k_y^2 V_1^2\right)} \right] = 0 , \quad (50)$$

where η and m_1 are given by Equations 43 and 45, respectively, and ϵ stands for the coefficient of wk^2 in Equation 25.

Equation 50 reduces to the dispersion formula obtained earlier (Reference 6) for a rotating configuration of two uniform *incompressible* fluids. It also gives the equations obtained by Vandervoort (References 7 and 8) for a nonrotating field-free configuration of superposed fluids (the lower fluid being incompressible), and for a uniform incompressible fluid topped by a stratified gaseous atmosphere.

The dispersion relation, Equation 50, is extremely unwieldy in the general case. Therefore we shall investigate only marginal instability $(n = 0)$. By putting $n = 0$ in Equation 50 and making use of Equations 43, 45, and 25 with n^2 equal to zero, after some simplifications

$$\left[gk^2(\rho_2 - \rho_1) + k_x^2 \rho_2 k V_2^2\right]^2 + \beta k_x^2 \rho_1 V_1^2 \left[gk^2(\rho_2 - \rho_1) + k_x^2 \rho_2 k V_2^2\right]$$

$$= k_x^2 \rho_1^2 V_1^2 \left[k^2 k_x^2 V_1^2 - \left(\beta + \frac{g}{c_1^2}\right) gk^2\right] . \quad (51)$$

For the case where $k_y = 0$ and $k_x = k$, from Equation 51,

$$k^2 \left[\rho_2^2 V_2^4 - \rho_1^2 V_1^4\right] + k\left[2g(\rho_2 - \rho_1)\rho_2 V_2^2 + \beta \rho_1 \rho_2 V_1^2 V_2^2\right]$$

$$+ g^2 \left[(\rho_2 - \rho_1)^2 + \frac{\rho_1^2 V_1^2}{c_1^2} + \frac{\beta}{g} \rho_1 \rho_2 V_1^2\right] = 0 . \quad (52)$$

If $H_1 = H_2 = H$ (no initial current sheet at $z = 0$) the following is the expression for the critical wave number:

$$k_* = \frac{g\beta\rho_2 + g^2 \left[\frac{\rho_1}{c_1^2} + \frac{4\pi(\rho_2 - \rho_1)^2}{\mu H^2}\right]}{2g(\rho_1 - \rho_2) - \beta \frac{\mu H^2}{4\pi}} , \quad (53)$$

where β is given by Equation 24. If the value of β from Equation 24 is used, then Equation 53 becomes

$$\frac{k_* c_1^2}{g} = \frac{1 + M_1 \left[(1 - \alpha)^2 - \dfrac{\alpha}{0.5 + M_1 \gamma^{-1}} \right]}{2(1 - \alpha) + \dfrac{1}{0.5 + M_1 \gamma^{-1}}} , \tag{54}$$

where $\alpha = \rho_2/\rho_1$ and $M_1 = c_1^2/V_1^2$.

If $\alpha = 1$ it follows from Equation 54 that k_* decreases linearly with an increase in M_1 (i.e., a decrease in the magnetic field), from $g/2c_1^2$ for $M_1 \to 0$ (i.e., an extremely high magnetic field) to zero for $M_1 = \gamma/2(\gamma - 1)$. Thus a configuration consisting of homogeneous liquid topped by a stratified atmosphere, so that no density discontinuity exists at the interface, is stable in the absence of a field and for small fields with $V^2/c^2 < 2(1 - \gamma^{-1})$. For fields exceeding that given by $V^2/c^2 = 2(1 - \gamma^{-1})$, the situation is unstable and only wavelengths greater than the critical value $\lambda_* (=2\pi/k_*)$ result in instability. The critical wavelength depends upon the existing field and tends to a finite value $\lambda_* = 4\pi c_1^2/g$ for infinitely high magnetic fields.

When $\alpha \neq 1$, it easily follows from Equation 54 that the configuration is thoroughly (for all k) stable for $\alpha > 1$, and thoroughly unstable for $\alpha < 1$, in the absence of magnetic field. For $\alpha < 1$ and in the limits of small and high magnetic fields, the configuration is unstable for $k < k_*$. If $\alpha > 2$, the situation is one of stability for high and low fields, whereas if $1 < \alpha \leq 2$, then $k_* = g/2(1 - 2\alpha)c^2$ for $M_1 = 0$ and the configuration, though stable for vanishingly small fields, becomes unstable in the limit of high magnetic fields.

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